

# Localization properties of one-dimensional speckle potentials in a box

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We investigate the localization properties of the single particle spectrum of a one-dimensional speckle potential. We consider both the repulsive and the attractive cases. The system is controlled by a couple of parameters: (i) a dimensionless variable, determined by the mass of the particle, the correlation length and the average intensity of the field; (ii) the size of the box containing the particle. Depending on the value of these parameters and the considered energy level, the eigenstates exhibit different regimes of localization. We compared our numerical results to the present theoretical framework, showing the key role played by the finite size of the system and the self-correlation of the potential when the observed behaviour turns out to be different from the predicted one.

## I. INTRODUCTION

Random potentials created by laser speckles are a popular tool employed in experiments with ultracold atoms for investigating the behavior of disordered systems [1]. In particular, one-dimensional (1D) speckle potentials have been the object of an intensive study in recent years. Many interesting features have been addressed from both the experimental and the theoretical sides, including classical localization and fragmentation effects, frequency shifts and damping of collective excitations, inhibition of transport properties, Anderson localization and related phenomena [2–10].

Despite this intense research activity, the properties of the single particle spectrum of the speckle potential have been addressed only partially, even in the 1D case. For the uniform case, the low energy behavior has been first considered in [7], where it has been shown that density of states of repulsive speckles is characterized by a Lifshitz tail [11, 12]. This has been discussed more thoroughly in [13], for both attractive and repulsive speckles. There it has also been shown that, as for the characteristic extension and the localisation properties of the eigenstates, three different regimes of speckle intensities  $s$  can be identified: semiclassical ( $s \gg 1$ ), intermediate, and quantum ( $|s| \leq 1$ ). In the latter case, the presence of an effective mobility edge - separating localized and extended eigenstates - has been discussed in [9], and its consequences have been observed in the Anderson localization experiment [10]. In addition, the existence of both extended and localized states in the presence of an inhomogeneous confinement has been discussed in [14].

What is lacking so far is a detailed numerical analysis of how the density profile of the eigenstates is related to the main properties of the potential (i.e.: spatial extension, intensity, blue or red detuning), depending on the chosen energy level.

In this article we consider a 1D speckle potential delimited by infinite walls - a sort of *disorder in a box* - and discuss the localisation properties of its eigenstates as a function of the speckle intensity  $s$  and the size  $L$  of the system. The box-like barriers can be realised experimentally by focusing two laser beams perpendicular

to the speckle direction, as discussed in [15], allowing for the investigation of finite size effects in a textbook case.

We studied both the ground state and the excited states, having the main target of characterizing the localisation properties. In the two cases we used specific observables, in order to explore the behaviour of the eigenstates while varying the parameters  $s$  (amplitude of the potential),  $L$  (size of the system) and the energy level  $E$ . When possible, we extrapolated the behaviour of the system in the limit  $L \rightarrow \infty$ .

We observed a stochastic translation of the center of mass of some eigenstates while varying  $s$ , depending on the single realisation of the speckle potential. Comparing a numerical measure of the probability of observing this phenomenon and a theoretical framework developed in this work, we separated the contribute linked to the boundary effects from the one depending only from the shape of the disordered potential. We expressed the latter as a function of the participation ratio.

According to the traditional literature, when  $L \rightarrow \infty$  a threshold energy level  $E_C(s)$  can be defined, that separates localised from extended states (corresponding to an effective mobility edge). Moreover, this theoretical framework gives a closed form expression for the localisation length  $\ell_{loc}$  as a function of  $E < E_C$  and  $s$  and a validity range for it. We performed closed form and numerical analysis of the validity range of the theory and restricted the interval of  $(s, E)$  where the predictions hold. Moreover, we described the behaviour of the system out of this new range in a closed form that fits our simulations.

Starting from the traditional tools of multifractal analysis, we defined a functional able to distinguish the localised eigenstates from extended ones and from the ones belonging to the crossover region of the spectrum. We verified that this observable improves the performance of the traditional ones and we used it in order to characterize the three regions of the spectrum while varying  $s$ .

The article is organized as follows. In Sec. II we define both the considered system and the main tools that we used to characterize its eigenstates. Then in Sec. III we discuss the properties of the ground state and the depen-

dence of its appearance from  $s$  and  $L$ . The density profile of the excited states is considered in Sec. IV. Our results are summarized in Sec. V.

## II. MODEL AND METHODS

Let us consider a single particle in a 1D speckle potential  $V_s(x) = V_0 v(x/\xi)$ , with intensity  $V_0 = \langle V_s \rangle$  and autocorrelation length  $\xi$  [5, 16].  $V_0$  can be both positive or negative, the potential resulting in a series of barriers or wells, that in the following will be referred as *repulsive* and *attractive*, respectively. The autocorrelation  $\xi$  represents a natural length scale for the system, and  $E_\xi = \hbar^2/2m\xi^2$  the corresponding energy scale. The hamiltonian of the system can be written in a dimensionless form as

$$H = -\frac{d^2}{dx^2} + sv(x), \quad (1)$$

where the rescaled intensity  $s = 2m\xi^2 V_0/\hbar^2$  represents the only parameter of the system. In case of a finite size system also the size  $L$  becomes relevant; its effects will be discussed hereinafter.

The speckle pattern can be generated numerically as discussed in [5] (and references therein). Then, the spectrum can be obtained by solving the stationary Schrödinger equation  $H\psi = E\psi$ , by mapping  $H$  on a grid with vanishing boundary conditions, corresponding to infinite walls.

For a fixed system size  $L$ , the localisation behaviour of the eigenstates in the different intensity regimes of speckle potential have been outlined in [13]. The characteristic extension and the localisation properties of the eigenstates depends considerably on  $s$ , and one can identify three different regimes, ranging from the semiclassical limit, for  $|s| \gg 1$ , down to the quantum regime, when  $|s| \leq 1$  (with an intermediate regime in-between). In the case of attractive speckles, and for shallow potentials (quantum regime), the eigenstates extend over several potential wells. The lowest lying eigenstates localise around deep wells but the ground state is not necessarily localised in the deepest well as the width of the well plays also a crucial role. When  $|s|$  is increased, the eigenstate width shrinks and their position may change. Eventually, for large enough  $|s|$  (semiclassical regime), the eigenstates tend to stack up as bound states inside isolated wells, with the ground state being the lowest eigenstate of the deepest well. A similar scenario holds for repulsive speckles, in this case the localisation taking place in between barriers.

Here we discuss it more thoroughly, characterizing the spectrum as a function of  $s$ , and discussing its dependence on  $L$ . In particular we consider a system length typically in the range  $L/\xi \in [100, 1000]$  (up to  $L = 7000$  in certain cases), and average over 50 to 500 speckle realisations (see Appendix C for further details).

In order to characterise the localisation properties of the eigenfunctions we use the following quantities.

1. The participation ratio

$$PR[\psi] = \frac{L}{\int_L dx |\psi(x)|^4} \quad (2)$$

that measures the relative extent occupied by the state  $\psi$ ; it decreases as  $\psi$  gets more and more localised.

2. The localisation length  $\ell_{loc}$ , that characterises the exponential decay of the tails of Anderson localised states,  $|\psi(x)|^2 \propto \exp\{-2(x-x_0)/\ell_{loc}\}$ , with  $x_0 \equiv \text{argmax} \{|\psi(x)|^2\}$  being the localization center. See Appendix C for further details on this quantity and the next two.

3. The average curvature on the tails of the wave function, defined through the following functional

$$\eta(s, L) = \frac{1}{2} \sum_{i=1}^2 \frac{1}{L_i} \left| \int_i \frac{d^2 \ln |\psi(x)|^2}{dx^2} dx \right|, \quad (3)$$

where  $L_i$  is the extent of left and right tails (over the ground noise).  $\eta$  vanishes for exponentially localised states, and is positive definite otherwise.

4. The *localization volume*  $D_{loc}$  defined as the length of the interval over which the exponential localization occurs.
5. The functional  $\sigma_\omega^2[\cdot]$  (defined in §IV B), that we introduce in order to characterize the eigenstates of the system by relating their localised/extended behaviour with their grade of self-similarity.

The first two are widely used in the literature, while the others are defined here in order to show specific features of this system that more commonly employed functionals are not effective in measuring.

In the next section we start by considering the properties of the ground state as a function of  $s$ , discussing their dependence on the finite size of the system. Then, in the following section we will consider the behaviour of the full spectrum.

## III. THE GROUND STATE

For a fixed system size  $L$ , the localization behavior of the eigenstates in the different intensity regimes of speckle potential have been outlined in [13]. Here we discuss it more thoroughly. We start by considering the density distribution of the ground state for a single realization of length  $L = 400$ , as a function of  $s$ , as shown in Fig. 1. As an example, we discuss the case of a red detuned speckle realization; a similar behavior is observed for the blue detuned case as well.

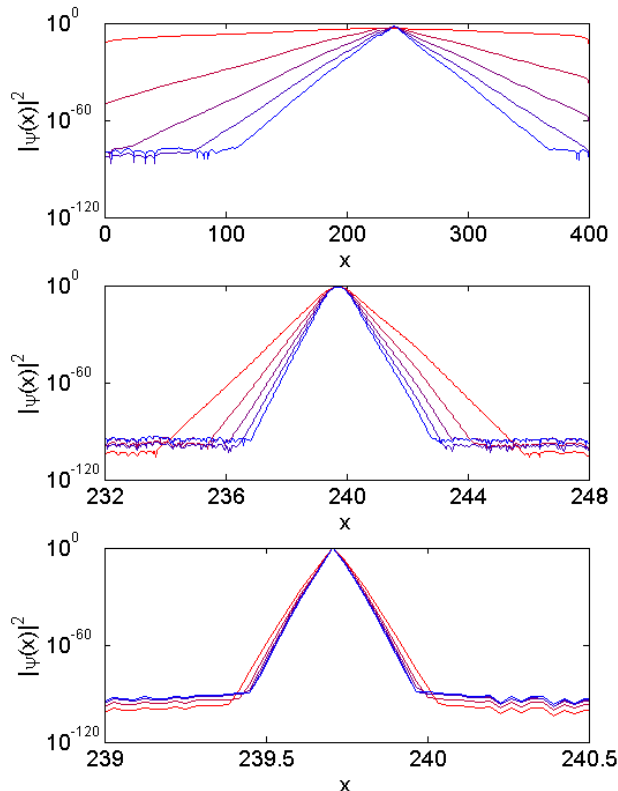


FIG. 1. Behaviour of the density  $|\psi(x)|^2$  of the ground state for a single realization of red-detuned speckle potential ( $L = 400$ ). We have considered the following intervals of  $|s|$ , respectively from the top to the bottom:  $2 \cdot 10^{-3} < |s| < 2 \cdot 10^{-1}$  (top);  $4 < |s| < 40$  (center);  $10^5 < |s| < 10^6$  (bottom). We plotted the considered values of  $|s|$  in increasing order from red to blue line.

According to our simulations, some of the behaviours observed in Fig. 1 have a general validity:

(i) For too small values of  $|s|$  the localization behavior of the ground state is affected by the boundaries. In this case the tails decay exponentially, but the finite size of the system limits the localization length  $\ell_{loc}$  to values of the order of the system size  $L$ .

(ii) In the opposite limit, for large values of  $|s|$  the ground state is bounded in a single well (red speckles) or between two adjacent barriers (blue speckles). In this case the tails ceases to decay exponentially.

(iii) For intermediate values of  $|s|$  the behavior of the tails continuously connects the two extreme regimes.

(iv) For some speckle realization, the localization center of the ground state may change as a function of  $s$ , see e.g. Fig. 2. This effect will be discussed in §III B.

Two of the quantities defined in §II -  $\eta(s)$  and  $D_{loc}(s)$  - allow for measuring the interval of  $|s|$  where there is a boundary effect (dark grey area), as shown in Fig. 3 for the same speckle realization used for Fig. 1. In fact  $D_{loc}(s)$  remains constant and  $\eta(s)$  decreases until  $|s| \gtrsim -10^{-2}$ , when the tails detach from the boundaries

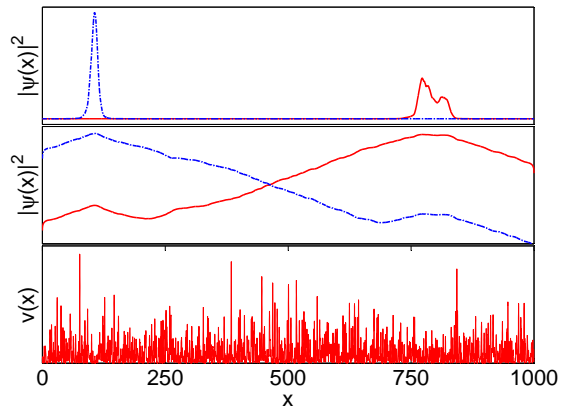


FIG. 2. An example of the change of localization position for increasing  $|s|$ . In the example we considered  $L = 1000$  and  $s \simeq 0.1$ . The two ground states are obtained with a shift  $\Delta s \simeq 10^{-7}$ .

of the system. The decrease of  $\eta(s)$  is linked to the local curvature of the tails when they are too near to the boundaries and, for even smaller  $|s|$  values, to the fact that the ground state cannot present a completely exponential localization. When the boundary effects are over (Fig. 3, light grey area), there is an interval of  $|s|$  where  $\eta(s) \simeq 0$ : a “pure” Anderson localization occurs. By increasing  $|s|$  we observe that the average curvature of the ground state begins to grow up, due to the increasing importance of the deep fluctuation of the potential where the speckle is localized. We have not observed any clear dependence of the amplitude of the region where a “pure” Anderson localization occurs from  $s$  and  $L$ , since there is a very strong dependence of  $\eta(s)$  from the shape of the single speckle potential, until it starts to constantly grow up. In fact, the most of the considered speckle realizations show a noisy behavior of  $\eta(s)$  for small  $|s|$  values, due to the presence of big local fluctuations of the potential and to random changes of the localization position.

However, we observed on average an abrupt rise of  $\eta(s)$  for  $|s|$  ranging from 1 to 3 (considering  $100 \leq L \leq 500$ ).

### A. Localization type

We discuss here how the localization properties of the states - either exponentially localized by the full random profile of the potential or bound states of single fluctuations (in a single well or between two barriers) - depend on  $s$  and  $L$ .

In the paragraph above we discussed the localization process of the ground state density  $|\psi(x)|^2$  at increasing  $|s|$ . Fig. 4 describes the same process averaged over 50 realizations of the speckle potential. We can observe that there is no transition from extended to localized states for both the attractive and the repulsive cases. In spite of this, both  $PR(s)$  and  $\ell_{LOC}(s)$  exhibit a “knee” at in-

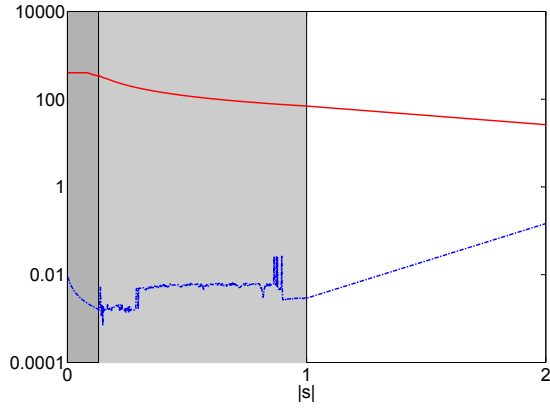


FIG. 3.  $\eta(s)$  (blue dashed curve) and  $D_{loc}(s)$  (red solid curve) for the ground state of the speckle realization considered in fig. 1.

creasing  $|s|$ . This knee separates the region where the localizing tails are influenced by the whole extension of the system (grey area) from the one where the leading contribute to the shape of  $|\psi(x)|^2$  is given by the local configuration of the potential around the localization point (white areas).

Though the knees are not exactly coincident for  $PR(s, L)$  and  $\ell_{LOC}(s, L)$ , their positions on  $s$  axis are comparable. The grey area extends conventionally from  $s \simeq -100$  to  $s \simeq 300$ : in fact the localization process is faster for  $s < 0$ , according with the intuitive fact that a deeper well is more effective than two higher next barrier, in containing the localized ground state at increasing  $|s|$ . Moreover, the dependence of  $PR(s, L)$  and  $\ell_{LOC}(s, L)$  on  $L$  is neglectable (though present). As we will discuss in §IV A, we will see that this holds only for the ground state case, since  $L$  plays a more relevant role for the degree of localization of the eigenstates while increasing  $E$ .

The curvature  $\eta(s, L)$  is not less sensible to  $s$  variations in the white areas than in the grey one. This is not surprising: even if an eigenfunction is so localized to be bounded in a single well, it goes on increasing its curvature as the depth of the well dilates. Moreover,  $\eta(s, L)$  exhibits a non neglectable dependence from  $L$  at  $|s| \gg 1$  values. This dependence from the extension of the system is related to a local effect: in fact, a greater  $L$  implies a bigger number of wells (couple of barriers) and, consequently, a deeper global minimum (higher couple of next maxima) on average, where the ground state can be bounded more effectively and so exhibit a greater  $\eta(s, L)$  value. The role of  $L$  is more relevant for  $s < 0$ , consistently with the greater binding effectiveness of the wells than the one of the barriers, as stated above.

In Fig. 5 we can observe the behaviour of  $D_{loc}(s, L)$  averaged over 50 realizations of the speckle potential. As stated before,  $D_{loc}(|s| \ll 1, L) \simeq L$ . However, when  $|s| \gtrsim 1$  the dependence from  $L$  disappear for both  $s \leq 0$ .

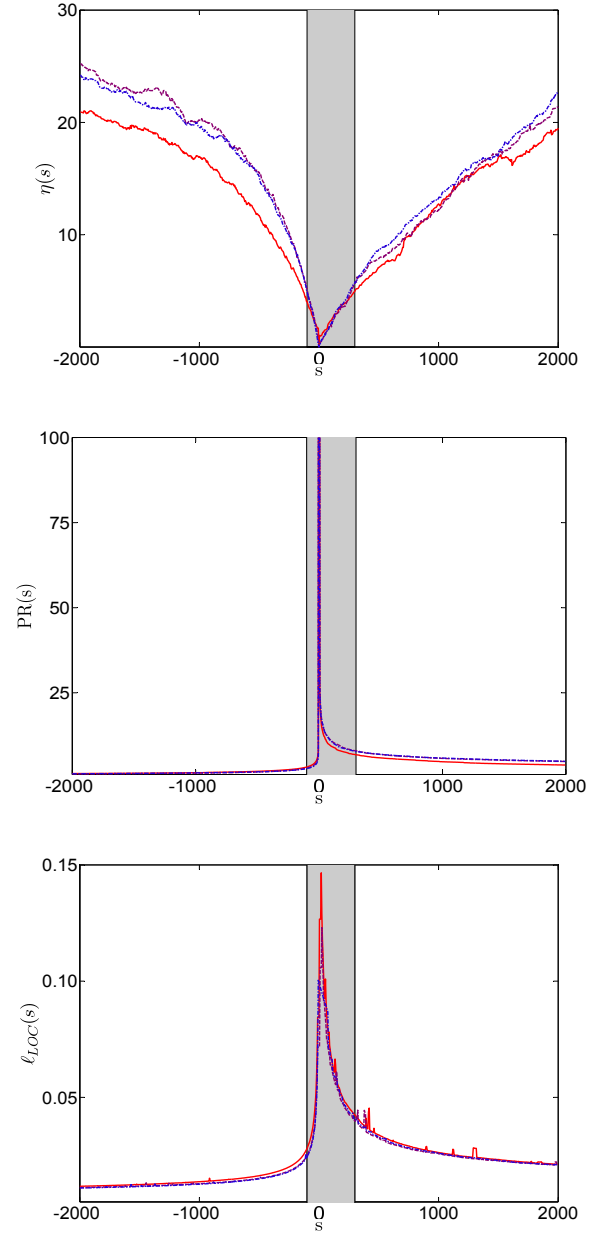


FIG. 4. The functionals  $\eta$  (top panel),  $PR$  (central panel) and  $\ell_{LOC}$  (bottom panel) for the ground state as a function of  $s$ , increasing the size  $L$  of the system from  $L = 100$  (red line) to  $L = 500$  (purple line) and  $L = 1000$  (blue line). For each choice of  $s, L$  considered, the plotted value is averaged over 50 realizations of the speckle potential.

## B. Variation of the localization position

In the following we analyze the phenomenon observed in figure 2. The change of localization position (CLP in the following) originates from the possibility that two next eigenstates exchange their position in the energy spectrum. This fact depends on the shape of the single realization of the potential and we found speckles without

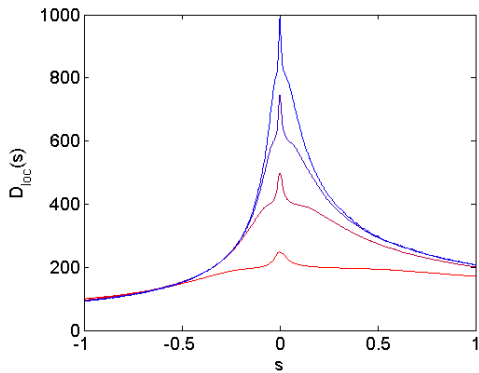


FIG. 5. The functional  $D_{loc}$  for the ground state as a function of  $s$ , increasing the size  $L$  of the system from  $L = 250$  (orange line) to  $L = 1000$  (blue line). For each choice of  $s, L$  considered, the plotted value of  $D_{loc}(s, L)$  is averaged over 50 realizations of the speckle potential.

CLP events on all the tested interval of  $s$ , as well as speckles which shows this behaviour one or more times.

Given a set of speckle realizations and considering for each potential only the first CLP event observed for the ground state – where possible – while increasing  $|s|$ , we build the cumulative density function  $P_{clp}(s, L)$ . The result is shown in figure 6.

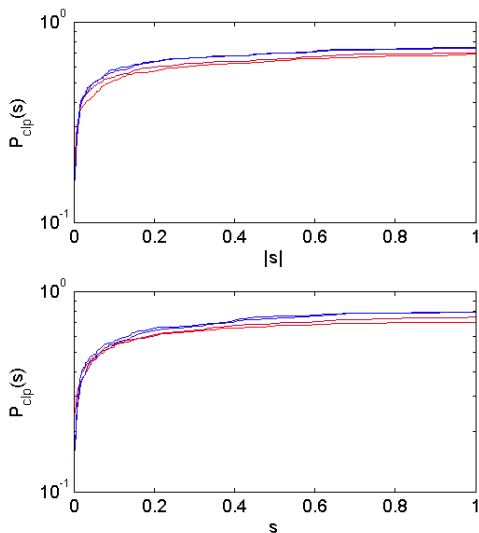


FIG. 6. Probability distribution of observing one or more *change of localization point* event for  $|s'|$  lesser than or equal to a given  $|s|$ . The function was obtained simulating the behaviour of the ground state over 500 speckle realizations as a function of  $|s|$ , for both red detuned (top) and blue detuned (bottom) speckle potential.

With the exception of a small interval of  $|s| \simeq 0$ , where the boundary effects play a major role in the energy level of the ground state, the probability density function  $p_{clp}$  of observing a CLP event can be linked with the parti-

cipation ratio  $PR(s, L)$  in a closed form. In fact we have

$$p_{clp}(s, L) = \sqrt{\frac{PR(s, L)}{2\pi}}, \quad (4)$$

This result is obtained in Appendix B. Our tests show that equation (4) holds with a good degree of approximation. In figure 7 we plotted the crossover between two regions: the one where the boundary effects prevail ( $s \simeq 0$ ) and equation (4) is not verified and the other (large enough  $|s|$ ) where the numerical simulation of  $p_{clp}$  is well described by equation (4).

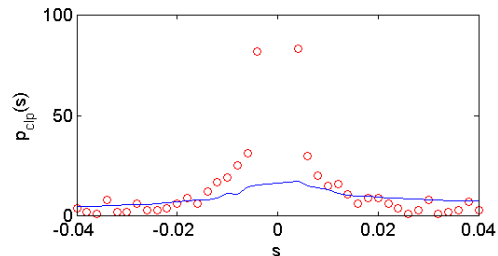


FIG. 7. Probability density function for the occurrence of a CLP event. Red circles are computed from the numerical results plotted in figure 6 for  $L = 1000$ ; the blue solid line is obtained from equation (4).

#### IV. EXCITED STATES

Let us now consider the behavior of the excited states as a function of their energy  $E$ . We expect the lower part of the spectrum to be affected mainly by the presence of the “disordered” potential, whereas at high energy the finite-size effect (the effect of the boundary conditions) should be predominant. As an example of the behaviours discussed in this paragraph, in Fig. 8 and 9 (top panels) we show a density plot of the eigenstates, for a single realization of the speckle potential, for both attractive and repulsive cases (the functional  $\sigma_\omega^2$  and the lower panels are discussed in §IV B). We observe the presence of three regions: localized states for low energy, extended states for high energy, separated by a wide intermediate region. In the following we analyze the localization properties of the eigenstates as a function of their energy  $E$ , by considering their multifractal behavior and localization length.

##### A. Localization length

In [9] is shown that, under the Born approximation, there is a high-momentum cutoff  $k_C$  (that represents an

effective ME), such that all the matter waves with momentum  $k > k_C$  are extended on every length scale (i.e.  $\ell_{LOC}^{-1}(k \geq k_C) = 0$ ).

In our case (see also [5, 13]) the expression for  $\ell_{LOC}$  given in [9] reads (see Appendix A)

$$\ell_{loc}(s, E) = \begin{cases} \frac{7.04}{s^2} \frac{E - s}{1 - (0.88\pi)^{-1}\sqrt{E - s}} & E < E_C \\ +\infty & E \geq E_C \end{cases} \quad (5)$$

with

$$E_C = s + (0.88\pi)^2. \quad (6)$$

and under the condition

$$|s| \ll s_C = 2(0.88\pi)^2. \quad (7)$$

The Mobility Edge is not directly observable using numerical techniques. In fact, an infinitely long tail requires an infinitely extended system in order to be free from boundary effects. However, from equation (5) we know that the following should be true

$$\lim_{L \rightarrow +\infty} \ell_{loc}(s, L, E_C(s)) = +\infty \quad (8)$$

for all  $s$  that satisfy equation (7). So, a numerical marker of the presence of a Mobility Edge is that  $\ell_{loc}(s, L, E_C)$  is increasing, as a function of  $L$ . In order to investigate the presence of this marker, we simulated the eigenstates of the system around  $E_C(s)$ , using different values of  $s$  and the largest values of  $L$  we can numerically access at the moment. As shown in figure 10, we can observe the following outcomes

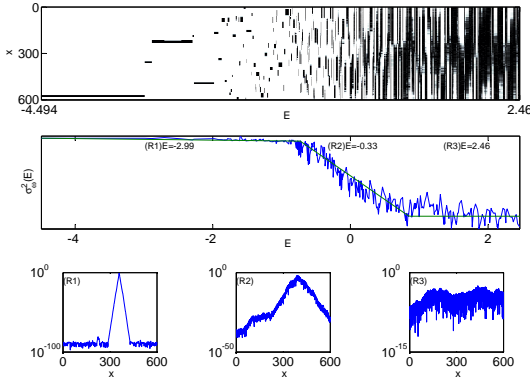


FIG. 8. (top) Example ( $s = 1$ ) of density plot for the spectrum of a red detuned speckle potential. (center) Comparison between the eigenstates  $\psi_E$  of a single realization of the speckle potential and  $\sigma_\omega^2[\psi_E]$ . This will be discussed in §IV B. (bottom) Density profiles of three eigenstates are plotted in a convenient logscale, in order to show the differences amongst the localized, delocalizing and delocalized wave functions.

- There is a range of values of  $s$  that verifies equation (8), suggesting the presence of an effective Mobility Edge in the limit  $L \rightarrow \infty$ .

- The boundaries of this range are not the same for the red-detuned and the blue-detuned potentials. Equation (7) becomes

$$s_C^{red} \lesssim s \lesssim s_C^{blue} \quad (9)$$

where  $-2 < s_C^{red} < -1$  and  $0.1 < s_C^{blue} < 1$ .  $s_C^{red}$  and  $s_C^{blue}$  have not to be intended as critical values of a transition between two regimes (Mobility Edge/other), because the condition to be satisfied is an approximation. Considering this, we can expect that the passage from the Mobility Edge occurs with a crossover.

- For both the attractive and the repulsive potential and for  $L \gg 1$ ,  $\ell_{loc}$  seems to be independent from  $L$ , considering  $L \gtrsim 5000$  and  $s \notin [s_C^{red}, s_C^{blue}]$ .

From figure 10 we can observe that  $\ell_{LOC}(s = \pm 0.1, L \geq 4000, E = E_C) \simeq L$ . This means that the eigenstates are extended for the tested values of  $L$ . However, the fact that the eigenstates go on being delocalized at  $E \simeq E_C$  for increasing values of  $L$  is compatible with equation (8).

For lesser values of  $L$  than the range tested above, we could measure  $\ell_{LOC}(E)$  starting from the ground state to  $E = E_C$  and beyond. Until  $\ell_{LOC}$  is comparable to  $L$ , we can test the prediction of equation (5) against the simulation being free from boundary effects.

As shown in figure 11, the numerical measure of  $\ell_{LOC}(s, E)$  fits the trend predicted in equation (5) only for  $s = \{\pm 0.1, -1\}$  (until the extension of the tails reach the boundaries of the system). The same values of  $s$  satisfy equation (8), confirming the range of validity estimated in equation (9). We have seen above that  $\ell_{LOC}$  becomes independent from the size of the system for large enough values of  $L$ , for blue-detuned potentials out of the

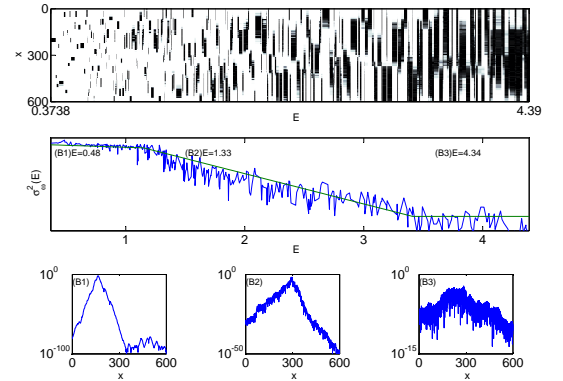


FIG. 9. This figure is analogous to Fig. 8 for a blue detuned speckle potential. In this case we have considered  $s = 1$ .

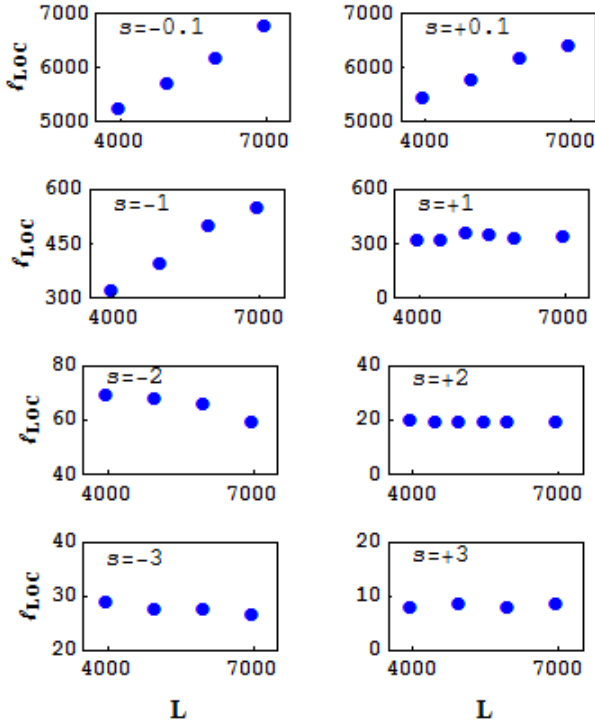


FIG. 10. Localization length  $\ell_{loc}(s, L, E)$  measured for  $L \in [4000, 7000]$  and  $E \in [E_C(s) - 0.25, E_C(s) + 0.25]$ . For each value of  $L$  and  $s$ ,  $\ell_{loc}$  is averaged over the range of  $E$  considered for 50 realizations.

range stated in equation (9). In the following, we study the approach to  $\frac{\partial}{\partial L} \ell_{LOC}(s, L, E) = 0$  for increasing  $L$ .

The figure 12 confirms the non comparability between the prediction of equation (5) and the numerically observed behaviour of  $\ell_{LOC}(s = 1, L, E)$ , considering  $L \in [200, 900]$ . In order to describe this behaviour, we followed the steps shown in figures 13 and 14:

1. We looked for the function of  $E$  which fits the simulations better, keeping  $s$  and  $L$  constant. As shown in figure 13, we have:

$$\ell_{LOC}(E)|_{s,L} = \exp\{g_1 \ln(E) + g_2\} \quad (10)$$

2. We looked for the functions of  $L$  which fit  $g_1$  and  $g_2$  better. As shown in figure 14, we have:

$$\begin{aligned} \hat{g}_1(L) &= \alpha_1 L / (\beta_1 + L) \\ \hat{g}_2(L) &= \alpha_2 / (\beta_2 + L) + \gamma_2 \end{aligned} \quad (11)$$

which imply the asymptotic independence of  $\ell_{LOC}$  from  $L$  in the limit  $L \rightarrow \infty$ .

Generalizing the results contained in equations (10) and (11) for all  $s \gtrsim s_C^{blue}$ , we can suppose that

$$\ell_{LOC}(s, L, E)_{s \gtrsim s_C^{blue}} \propto E^{h(s)} \quad (12)$$

for  $L \gg 1$  (expressed in  $\xi$  units). Testing several  $s$

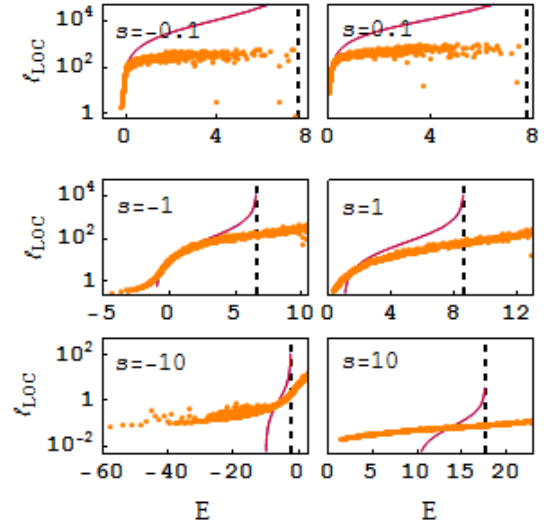


FIG. 11.  $\ell_{LOC}(E)$  for  $-10 \leq s \leq 10$  and  $L = 800$ . The figure shows a comparison between the simulations (orange dots, averaged over 50 realizations of the potential) and the theoretical results for  $L = +\infty$  (continuous line).

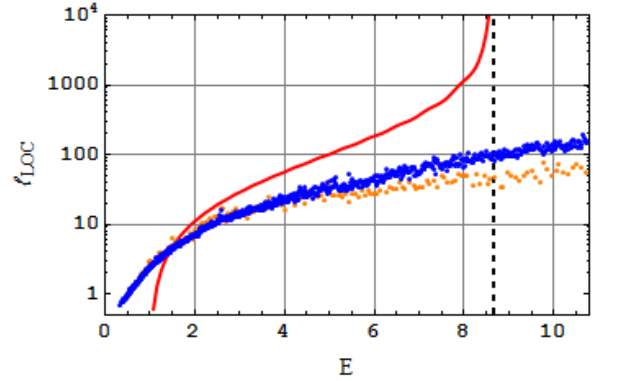


FIG. 12.  $\ell_{LOC}(E)$  for  $s = 1$  and  $L = 200$  (orange dots),  $L = 900$  (blue dots). The figure shows a comparison between the numerical simulations (dots, averaged over 50 realizations of the potential) and the theoretical prediction (continuous line). The dashed line marks  $E_C$ .

values, we found that the function  $h(s)$  in equation (12) which best fits the results of our numerical simulations is  $h(s) = h_1 |s|^{h_2}$ . The same approach described above can be applied to the red-detuned case, leading to the same results. Finally we have:

$$\ell_{LOC}(s, L, E) = h_0 E^{h_1 |s|^{h_2}} \quad (13)$$

Eq. (13) holds for  $L \gtrsim 5000$ ,  $s \notin [s_C^{red}, s_C^{blue}]$  and  $E > \max(0, E_{gs})$ . Table IV A reports an estimate of the parameters  $h_0$ ,  $h_1$  and  $h_2$ .



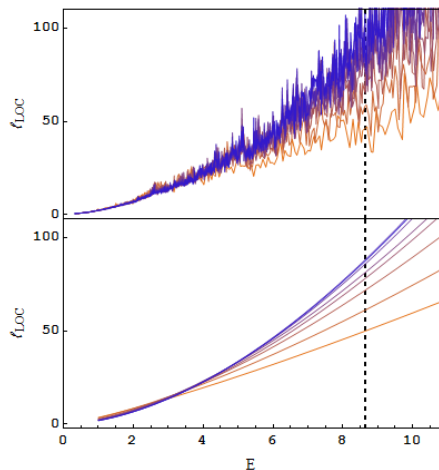


FIG. 13.  $\ell_{LOC}(E)$  for  $s = 1$  and  $L$  between 200 (orange) and 900 (blue). The top figure shows the numerical dependence on  $L$  while increasing  $E$ .  $\ell_{LOC}$  does not show any sensibility to the ME (dashed line). The numerical results are averaged over several realizations of the potential. The bottom figure shows a fit of the same output, using the function  $\hat{\ell}_{LOC}(E, L) = \exp\{g_1(L) \ln(E) + g_2(L)\}$  (chosen on the basis of the numerical evidence). The average value of the  $R^2$  of the fit is 0.985 across the fitted functions.

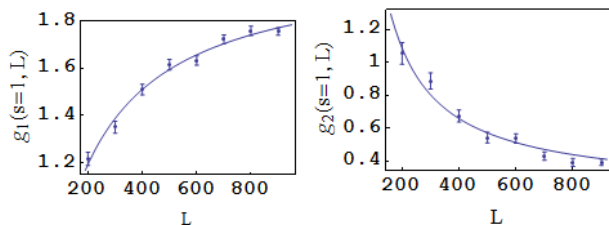


FIG. 14. The figure shows the fitted values of  $g_1$  and  $g_2$  (dots) across the tested values of  $L$ , plotted together their respective fit errors (bars), considering  $s = 1$  as an example. The numerical results are fitted using the functions described in equations (10) and (11).

### B. Self similarity of eigenstates as a marker of the delocalization

For finite size systems with a  $\delta$ -correlated disordered potential, the presence of a transition from localized to extended states has been found both for the three-

$sign(s)$	$h_0$	$h_1$	$h_2$
+	0.590	2.927	-0.901
-	5.048	2.209	-0.706

TABLE I. Best estimate of the parameters used in Eq. 13.

dimensional [17] and the one-dimensional case [18] (the so-called *Anderson transition*). For these systems, the eigenstates around the critical energy threshold show a kind of scale invariance known as multifractality.

Generally speaking, a scale invariance property of a given function  $f(x)$  can be numerically verified partitioning it into clusters of increasing size. The necessary condition that  $f(x)$  must satisfy for being considered as a fractal (or a multifractal) is the independence of a certain quantity  $\omega(q, \Lambda)$  - defined later on - from the chosen size  $\Lambda$  of the clusters. In the following we specialize  $f(x) = |\psi(x)|^2$ . Let us introduce the measures

$$\mu_k(\Lambda) = \int_{k\Lambda}^{(k+1)\Lambda} dx f(x) \quad k = 0, \dots, k_{max}(\Lambda) \quad (14)$$

where the interval  $[0, (k_{max} + 1)\Lambda \equiv L]$  is the spatial domain of the system for every  $k_{max} \in \mathbb{N}$ . For each  $q \in \mathbb{Z}$  and for each acceptable value of  $\Lambda$ , we can compute the  $q$ -th moment  $P_q(\Lambda)$ :

$$P_q(\Lambda) = \sum_k [\mu_k(\Lambda)]^q, \quad q \in \mathbb{Z}. \quad (15)$$

For a self-similar function, it holds that

$$P_q(\Lambda) \propto \Lambda^{\omega(q)}. \quad (16)$$

When the relation (16) is verified, the shape of  $\omega(q)$  defines the type of self-similarity of the function (if  $\omega(q)$  is linear with a single slope,  $f(x)$  is fractal; if  $\omega(q)$  is linear with two different slopes for positive and negative  $q$ ,  $f(x)$  is multifractal; otherwise  $f(x)$  is not a fractal in any sense). For every  $f(x)$  (even for the non self similar ones), the following quantity is well defined:

$$\omega(q, \Lambda) = \ln P_q(\Lambda) / \ln \Lambda. \quad (17)$$

Given that  $f(x)$  can be a (multi)fractal *only if* the numerical dependence of  $\omega(q, \Lambda)$  from  $\Lambda$  is almost absent, we can introduce a quantity to measure *how much far* is  $f(x)$  from being self-similar in some sense:

$$\sigma_\omega^2[f] = \sum_q \left\langle \omega^2(q, \Lambda) - \langle \omega(q, \Lambda) \rangle_\Lambda^2 \right\rangle_\Lambda. \quad (18)$$

The results obtained in [18] suggest that, if we can ignore the effect of the correlation in the speckle potential, our system should show an Anderson transition at a critical energy  $E_C$ , with a multifractal behaviour of the eigenstates with eigenvalues near to  $E_C$ . This is in agreement with the prediction of a ME in §IV A and it should lead to observe numerically

$$\lim_{E \rightarrow E_C} \sigma_\omega^2 [|\psi_E|^2] = 0 \quad (19)$$

We notice that both the predictions of a ME stated in §IV A and in this paragraph do not take into account the effect of the spatial correlation of the speckles at high



orders. In particular, the presence of a multifractal behaviour has been observed in completely decorrelated 1D system.

Looking for a quantity able to distinguish among different degree of localization, we compared the results obtained with  $\sigma_\omega^2[\psi_E]$  against the other main observables defined in §II and the variance of the density  $\sigma^2[\psi_E]$ . This functional turns out to be the only one able to react at the crossover between localized and extended states.

The link between the possible density profile of the wavefunctions and the correspondent values assumed by  $\sigma_\omega^2[\psi_E]$  is shown in Fig. 8 and 9. For both localized and extended states,  $\sigma_\omega^2[\psi_E]$  oscillates randomly around a constant value. In the intermediate delocalization region of the spectrum,  $\sigma_\omega^2[\psi_E]$  decreases while  $E$  is growing. In order to measure  $E_C^{(a)}$  and  $E_C^{(d)}$  (respectively the attachment point and the detachment point of the intermediate energy tranche where the delocalization process occurs), we fitted  $\ln \sigma_\omega^2[\psi_E]$  with a polygonal chain (see Appendix C for further details). The behavior of  $E_C^{(a)}$  and  $E_C^{(d)}$  as a function of  $s$  is shown in Fig. 15. We have also checked that the extent of the intermediate region is not affected by the system size  $L$  for small enough systems ( $L \lesssim 10^3$ ).

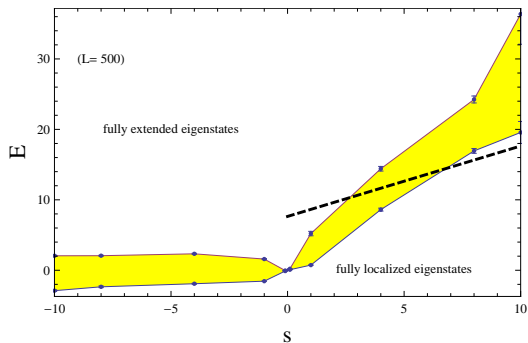


FIG. 15. The amplitude of the intermediate region as a function of  $s$ .  $E_C^{(a)}$  and  $E_C^{(d)}$  are measured averaging the values obtained from eq. (C1) over 50 realizations. The dashed line plots the predicted threshold for the Mobility Edge.

Fig. 15 shows the localizing behavior as a function of  $s$ :

- For  $s > 0$ , the intermediate region is shifted towards higher energy levels and its amplitude increases, for increasing values of  $s$ . This fact can be related to the distribution of the maxima of the potential, that are higher and more dispersed for larger values of  $s$ . Conversely, for  $s < 0$  the intermediate region is far more stable in mean and amplitude, according with a distribution of the maxima less sensibles to the  $s$  variations and more concentrate around its mean value.
- The prediction for the effective ME  $E_C(s)$  in eq. (6) is not in agreement with the position of the *intermediate* region  $[E_C^{(a)}, E_C^{(b)}]$ . This is coherent with

the results discussed in §IV A, where we verify that the finiteness of the system forbid the numerical observation of the predicted Mobility Edge.

- Eq. (19) is not verified. This can be related to the presence of self-correlation in our potential (unlike the case studied in [18]). Once again, nothing suggests the presence of an Anderson transition for the considered system.

The last two points suggest that the finite size of the system and the correlations in the disorder play a crucial role in the properties of the spectrum generated by a speckle potential and they cannot be neglected at any order.

## V. SUMMARY

In this work we focused on the density profile of the eigenfunctions associated to a particle in a *box-plus-speckle-potential* system. We analyzed how the parameters  $s$ ,  $L$  and  $E$  influence the density profile of the eigenfunctions, giving special attention to the degree of localization.

The ground state exhibits a very poor dependence from  $L$ . The passage from Anderson localized states to eigenstates bounded in a single well (or couple of barriers) is smooth at increasing  $|s|$  but it is still possible to partition the localization process into two distinct regimes. The localization is faster (as a function of  $|s|$ ) for the attractive case.

We found that the lower energy eigenstates can exhibit an abrupt shift of the localization point (and so of the center of mass of the system) while tuning  $|s|$ , due to an occasional degeneracy with the next energy level. We linked the probability of observing this effect to the Participation Ratio of the eigenstate and to the finite extension of the system - the latter contribute being relevant only for  $|s| \ll 1$ .

The system does not exhibit any Mobility Edge nor multifractal behaviour while delocalizing at increasing  $E$ . The first fact is attributable to the finite extension of the system, while the second is linked to the self-correlation of the potential. We described in a simple closed form the asymptotical behaviour of the localization length as a function of  $s$  and  $E$ , for  $L \gg 1$  and for  $s$  out of the validity interval of the mobility edge framework. Finally, we found that the eigenstates of this system at increasing energies can be divided in three distinct regions according to their degree of localization. The quantity developed here in order to observe this partition could find an application also in other contexts as an indicator of the degree of self-similarity in data series.

## VI. ACKNOWLEDGEMENTS

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### Appendix A: Generalization of the ME to finite-size systems

As stated in [9], the Born approximation in this context assumes the inequality

$$V_0 \tilde{\xi} \ll (\hbar^2 k/m)(k\tilde{\xi})^{1/2}, \quad (\text{A1})$$

where  $\xi = 0.88\pi\tilde{\xi}$ . Using in Eq. (A1) the definition of  $s$  given in Section II, we obtain the following equivalent condition:

$$E \gg s + Cs^{4/3}, \quad (\text{A2})$$

where  $C = 2^{-4/3}(0.88\pi)^{-2/3} \simeq 0.2$ . Under this condition, we have

$$\ell_{LOC}^{-1}(k) = \frac{\pi m^2 V_0^2 \tilde{\xi}}{2\hbar^4 k^2} f(k) \quad (\text{A3})$$

where  $\sqrt{\pi/2}f(k) = \sqrt{\pi/2}(1-k\tilde{\xi})\Theta(1-k\tilde{\xi})$  is the autocorrelation function of the potential in case of an infinitely extended system. Substituting the definition of  $s$  again, Eq. (A3) in our framework becomes

$$\ell_{LOC}^{-1}(s, E) = \frac{s^2}{8 \cdot 0.88} \frac{1 - \frac{\sqrt{E-s}}{0.88\pi}}{E-s} \Theta(s + (0.88\pi)^2 - E) \quad (\text{A4})$$

which leads to eq. (5). From eq. (A2) and (A4), it follows that

$$s + Cs^{4/3} \ll E < s + (0.88\pi)^2, \quad (\text{A5})$$

that leads to a stronger requirement than (A1) for allowing the good definition of  $\ell_{LOC}$ , as stated in eq. (7).

### Appendix B: CLP effect

In the following we are going to prove that the result in equation (4) holds except for a small interval around  $s = 0$ . Naively speaking, a change of localization point event (CLP event in the following) occurs when two next localized eigenstates exchange their positions in the spectrum because of a variation of  $s$ . Without loss of generality, let us consider the ground state  $\psi_0$  and the first excited state  $\psi_1$ , with their respective energy eigenvalues  $E_0$  and  $E_1$ . So we have

$$p_{CLP}(s) = p(E_0 = E_1 | s). \quad (\text{B1})$$

As shown before, low energy eigenstates of the discussed model are well described by

$$\psi_i(x) = c \left[ e^{-\frac{|x-x_{LOC}|}{\ell_{LOC}}} 1_{x \in D_{tails}} + g(x) 1_{x \notin D_{tails}} \right] \quad (\text{B2})$$

where  $D_{tails}$  is the domain where the wavefunction exponentially decays and  $g(x)$  is the ground noise out from  $D_{tails}$ . We want to explicit the value of  $E_i$ , in order to solve the equation (B1). We can state that

$$E_i = K_i + sV_i \quad (\text{B3})$$

using the fact that the hamiltonian is separable. For the kinetic term, approximating  $g(x) \simeq 0 \simeq e^{-\frac{|x-x_{LOC}|}{\ell_{LOC}}}$ , we have

$$\begin{aligned} K_i &= \int_0^L \psi_i(x) \frac{\partial^2}{\partial x^2} \psi_i(x) dx \\ &\simeq \frac{c^2}{\ell_{LOC}^2} \int_0^L e^{-2\frac{|x-x_{LOC}|}{\ell_{LOC}}} dx \\ &\simeq \frac{1}{\ell_{LOC}^2} \end{aligned} \quad (\text{B4})$$

this approximation holds for  $|s|$  large enough that boundary effects are neglectable. For the potential term we have

$$V_i = \int_0^L |\psi_i(x)|^2 v(x) dx \simeq \sum_j \psi_{ij}^2 v_j \quad (\text{B5})$$

where we approximate the continuous system applying an arbitrary discretization. Neglecting the self correlation of the potential, we can consider all the  $v_j$  as i.i.d. stochastic variables. Since  $v_j$  follows a negative exponential distribution, we have

$$\psi_{ij}^2 v_j \sim \Theta(v_j) \psi_{ij}^{-2} e^{-\psi_{ij}^{-2} v_j}. \quad (\text{B6})$$

A sum of independent exponentially distributed variables with different mean life values fits the generalized Erlang distribution. So we are able to determine its variance in a closed form.

$$\sigma^2[V_i] \simeq \sum_j \psi_{ij}^4 \simeq PR(s, L)^{-1} \quad (\text{B7})$$

On average, it holds that  $\sigma^2[V_0] = \sigma^2[V_1]$  and  $\mu[V_0] = \mu[V_1]$ . Approximating the distributions of  $V_0$  and  $V_1$  with two normal distributions, we have

$$\frac{E_0 - E_1}{s} = V_0 - V_1 := \tilde{v} \sim \sqrt{\frac{PR(s, L)}{2\pi}} e^{-\frac{PR(s, L)\tilde{v}^2}{2}} \quad (\text{B8})$$

Replacing equations (B3), (B4) and (B8) into equation (B1), we obtain

$$p_{CLP}(s, L) = p(\tilde{v} = 0 | s, L) = \sqrt{\frac{PR(s, L)}{2\pi}} \quad (\text{B9})$$

that leads to equation (4).

### Appendix C: Numerical methods and criteria

In the following section we summarize some methods that we used to obtain the results contained in Sec. III and IV.

*Generation of the eigenstates.* We use the following prescription:  $H_{ij} = 1/\Delta\tilde{x}^2 (2\delta_{ij} - \delta_{i+1,j} - \delta_{i-1,j}) + s\tilde{I}(j\Delta\tilde{x})\delta_{ij}$ , with vanishing boundary conditions,  $\psi_0 = \psi_{N+1} = 0$ . The number of grid points  $N$  is chosen so that every fluctuation of the potential is sampled with at least 8 points (i.e.:  $N \geq 8L/\xi$ ).

*Identification and quality check of the exponential tails.* We need to identify four  $x$  values ( $x_1, \dots, x_4$  in ascending order) in order to select the two tails properly. The routine starts choosing the points  $x_2$  and  $x_3$ , near the localization point  $x_{LOC} = \text{argmax} [|\psi(x)|]$  but far enough to exclude the local fluctuation of the potential around which the eigenfunction is localized (i.e.:  $x_{2,3} = x_{LOC} \pm \xi$ ).

The terminal point of the right tail  $x_4$  is found going through the  $|\psi(x)|$  values for increasing  $x > x_3$ , until  $|\psi(x)| \leq \psi_{noise}$  or  $x \geq L - \xi$ . The routine identifies  $x_1$  using a symmetrical rule to the one described above. The threshold  $\psi_{noise}$  has the form

$$\psi_{noise}(\alpha) := e^{[\alpha \max(\ln |\psi(x)|) + (1-\alpha) \min(\ln |\psi(x)|)]}$$

where  $\alpha$  is a user defined parameter.

We checked the exponential shape of the tails systematically, computing the  $R^2$  for the fit used to measure  $\ell_{LOC}$ . When  $R^2$  value returned an ambiguous indication, we verified the shape of the tails by a direct observation.

*Check for the physical meaning of the ground noise.* In § II we introduced the localization volume  $D_{LOC}$  to better understand the local influence of the boundaries

over the density profile.  $D_{LOC}$  is well defined only if the abrupt interruption of the exponential decay of the eigenfunctions out of the interval  $[x_1, x_4]$  is not a numerical effect - at least in the discussed region of  $s, L$ . We verified this fact performing the three tests described below over a sample of speckle realizations: (i) computation of the same eigenstate using different computers; (ii) computation of the same eigenstate using a re-sampled version of the speckle potential with a different  $N$  (produced by a spline interpolation); (iii) comparison of the stability of  $\psi(x)$  to an “adjusted version” of the same eigenstate  $\tilde{\psi}(x)$ . The latter is obtained extrapolating the exponential decay outside  $[x_1, x_4]$  until the machine double epsilon ( $\sim 5 \cdot 10^{-324}$  in this case) or a boundary of the system is reached. This test was performed applying the following condition to check the considered eigenstates

$$\left| \left[ \mathbf{1} - \hat{n} \circ \hat{\mathcal{H}} \right] \psi_E \right|^2 < \left| \left[ \mathbf{1} - \hat{n} \circ \hat{\mathcal{H}} \right] \tilde{\psi}_E \right|^2$$

where  $\hat{n} [v \in \mathbb{R}^N] := v/|v|$ . Ideally it should hold that  $\hat{n} \circ \hat{\mathcal{H}} \psi_E = \hat{\mathcal{H}} \psi_E / E$ , but we preferred the criterion above to avoid the error over the estimate of the eigenvalue  $E$ .

*Measure of  $E_C^{(a)}$  and  $E_C^{(d)}$  in §IV B.* The results shown in Fig. 8, 9 and 15 are obtained fitting  $\ln [\sigma^2] (E)$  with a polygonal chain. More in detail, we used the following function

$$\begin{aligned} \hat{\sigma}_\omega^2 [\psi_E] = & \exp \{ (c_1 E + c_2) \Theta(E_C^{(a)} - E) \Theta(E - E_{gs}) \\ & + (c_3 E + c_4) \Theta(E - E_C^{(a)}) \Theta(E_C^{(d)} - E) \\ & + c_5 \Theta(E - E_C^{(d)}) \} \end{aligned}$$

from which we obtained an estimate of  $E_C^{(a)}$  and  $E_C^{(d)}$ , respectively the attachment and the detachment point of the crossover region.

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